AN ESTIMATION FOR THE ESSENTIAL NORM OF COMPOSITION OPERATORS ACTING ON BLOCH-TYPE SPACES

JULIO C. RAMOS FERNÁNDEZ

Abstract. Let μ be any weight function defined on the unit disk $\mathbb{D}$ and let ϕ be an analytic self-map of $\mathbb{D}$. In the present paper we show that the essential norm of composition operator C_ϕ mapping from the α-Bloch space, with $\alpha > 0$, to μ-Bloch space $\mathcal{B}^\mu$ is comparable to

$$\limsup_{|a|\to 1^-} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu},$$

where, for $a \in \mathbb{D}$, σ_a is a certain special function in α-Bloch space.

Keywords: Bloch spaces, Composition operators.
MSC 2010: 30D45, 32A30, 47B33.

1. Introduction

Let μ be denotes what we call a *weight* on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$; that is, μ is a bounded, continuous and strictly positive function defined on $\mathbb{D}$, and let $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{D}$, which is equipped with the topology of uniform convergence on compact subsets of $\mathbb{D}$. The *μ-Bloch space* $\mathcal{B}^\mu(\mathbb{D})$, which we denote more briefly by $\mathcal{B}^\mu$, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_\mu := \sup_{z\in\mathbb{D}} \mu(z) \left|f'(z)\right| < \infty.$$

μ-Bloch spaces are called *weighted Bloch spaces*. For weights μ on $\mathbb{D}$, a Banach space structure on $\mathcal{B}^\mu$ arises if it is given the norm

$$\|f\|_{\mathcal{B}^\mu} := |f(0)| + ||f||_\mu.$$

These Banach spaces provide a natural setting in which one can study properties of various operators. For instance, Attele in [1] proved that if $\mu_1(z) := w(z)\log\frac{2}{w(z)}$, where $w(z) := 1 - |z|^2$ and $z \in \mathbb{D}$, then the Hankel operator H_f induced by a function f in the Bergman space $A^2(\mathbb{D})$ (see [4, Ch. 2]) is bounded if and only if $f \in B^{\mu_1}$, thus giving one reason, and not the only reason, why log-Bloch-type spaces are of interest. When $\mu(z) = v_\alpha(z) := \left(1 - |z|^2\right)^\alpha$ with $\alpha > 0$ fixed, then we get back the α-Bloch space which is denoted as $\mathcal{B}^\alpha$ and when $\alpha = 1$ we obtain the Bloch space $\mathcal{B}$.

A holomorphic function ϕ from the unit disk $\mathbb{D}$ into itself induces a linear operator C_ϕ, defined by $C_\phi(f) = f \circ \phi$, where $f \in H(\mathbb{D})$. C_ϕ is called the

composition operator with *symbol* ϕ. Composition operators continue to be widely studied on many subspaces of $H(\mathbb{D})$ and particularly in Bloch-type spaces.

The study of the properties of composition operators on Bloch-type spaces began with the celebrated work of Madigan and Matheson in [8], where they characterized the continuity and compactness of composition operators acting on the Bloch space $\mathcal{B}$. Many extensions of the Madigan and Matheson's results have appeared (see for instance [12] and a lot of references therein). In particular, Xiao in [17] has extended the results by Madigan and Matheson in [8] to composition operators C_ϕ acting between α-Bloch spaces. Recently, many authors have found new criteria for the continuity and compactness of composition operators acting on Bloch-type spaces en terms of the n-th power of the symbol ϕ and the norm of the n-th power of the identity function on $\mathbb{D}$. The first result of this kind appears in 2009 and it is due to Wulan, Zheng, and Zhu ([16]), in turn, their result was extended to α-Bloch spaces by Zhao in [19]. Another criterion for the continuity and compactness of composition operators on Bloch space is due to Tjani in [14] (see also [15] or more recently [16]), she showed the following result:

Theorem 1.1 ([14]). *The composition operator C_ϕ is compact on $\mathcal{B}$ if and only if $\phi \in \mathcal{B}$ and*

$$\lim_{|a|\to 1^-} \|\varphi_a \circ \phi\|_{\mathcal{B}} = 0,$$

where φ_a is a Möbius transformation from the unit disk onto itself; that is, $\varphi_a(z) = (a-z)\,/\,(1-\overline{a}z)$, with $z \in \mathbb{D}$.

This last result has been recently extended to α-Bloch spaces by Malavé and Ramos-Fernández in [9].

The essential norm of a continuous linear operator T between normed linear spaces X and Y is its distance from the compact operators; that is, $\|T\|_e^{X\to Y} = \inf\left\{\|T-K\|^{X\to Y} : K : X \to Y \text{ is compact}\right\}$, where $\|\cdot\|^{X\to Y}$ denotes the operator norm. Notice that $\|T\|_e^{X\to Y} = 0$ if and only if T is compact, so that estimates on $\|T\|_e^{X\to Y}$ lead to conditions for T to be compact. The essential norm of a composition operator on $\mathcal{B}$ was calculated by A. Montes-Rodríguez in [10]. He obtained similar results for essential norms of weighted composition operators between weighted Banach spaces of analytic functions in [11]. Other results in this direction appear in the paper by Contreras and Hernández-Díaz in [3]; in particular, formulas for the essential norm of weighted composition operators on the α-Bloch spaces of $\mathbb{B}_n$ were obtained (see also the paper of MacCluer and Zhao [7]). Recently, many extensions of the above results have appeared in the literature; for instance, the reader is referred to the paper of Yang and Zhou [18] and several references therein. Zhao in [19] gave a formula for the essential norm of $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ in terms of an expression involving norms of powers of ϕ.

More precisely, he showed that

$$\|C_\phi\|_e^{\mathcal{B}^\alpha\to\mathcal{B}^\beta} = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{j\to\infty} j^{\alpha-1}\left\|\phi^j\right\|_{\mathcal{B}^\beta}.$$

It follows from the discussion at the beginning of this paragraph that $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if

$$\lim_{j\to\infty} j^{\alpha-1}\left\|\phi^j\right\|_{\mathcal{B}^\beta} = 0.$$

The Zhao's results in [19] have been extended recently to the weighted Bloch spaces by Castillo, Clahane, Farías and Ramos-Fernández in [2]. Also, Hyvärinen, Kemppainen, Lindström, Rautio and Saukko in [6] obtained necessary and sufficient conditions for boundedness and an expression characterizing the essential norm of a weighted composition operator between general weighted Bloch spaces $\mathcal{B}^\mu$, under the technical requirements that μ is radial, and that it is non-increasing and tends to zero toward the boundary of $\mathbb{D}$.

The goal of the present paper is to give an estimate of the essential norm of composition C_ϕ mapping from $\mathcal{B}^\alpha$ to $\mathcal{B}^\mu$ which implies Theorem 1.1 and the result given by Malavé and Ramos-Fernández in [9]. More precisely, in the next section we will show the following result.

Theorem 1.2. *Let ϕ be an analytic self-map of the unit disk $\mathbb{D}$. Then for the essential norm of the composition operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\mu$ we have*

$$\|C_\phi\|_e^{\mathcal{B}^\alpha\to\mathcal{B}^\mu} \sim \limsup_{|a|\to 1^-} \|\sigma_a\circ\phi\|_{\mathcal{B}^\mu}. \tag{1}$$

The relation (1) means that there is a positive constant M_α, depending only on α, such that

$$\frac{1}{M_\alpha}\|C_\phi\|_e^{\mathcal{B}^\alpha\to\mathcal{B}^\mu} \le \limsup_{|a|\to 1^-} \|\sigma_a\circ\phi\|_{\mathcal{B}^\mu} \le M_\alpha\, \|C_\phi\|_e^{\mathcal{B}^\alpha\to\mathcal{B}^\mu}$$

and the functions σ_a with $a \in \mathbb{D}$ will be defined at the begin of the next section.

2. Proof of Theorem 1.2

The key to our results lies in considering the following family of functions. For $a \in \mathbb{D}$ fixed, we define

$$\sigma_a(z) = (1-|a|)\left((1-\overline{a}z)^{-\alpha} - 1\right), \qquad (z\in\mathbb{D}).$$

Clearly, for each $a \in \mathbb{D}$, the function σ_a has bounded derivative and for this reason we have that $\sigma_a \in \mathcal{B}^\alpha$. In fact, it is easy to see that

$$\sup_{a\in\mathbb{D}} \|\sigma_a\|_{\mathcal{B}^\alpha} \le \alpha\, 2^\alpha.$$

Furthermore, it is clear that if $\frac{1}{2} < |a| < 1$, then

$$\left|\sigma_a'(a)\right| \geq \frac{\alpha}{4\left(1-|a|^2\right)^{\alpha}}. \tag{2}$$

Also, we can see that σ_a goes to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1^-$. Also, we will need the following lemma which is well known and is consequence of a more general result due to Tjani in [13]:

Lemma 2.1. *The composition operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\mu$ is compact if and only if given a bounded sequence $\{f_n\}$ in $\mathcal{B}^\alpha$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|C_\phi(f_n)\|_{\mathcal{B}^\mu} \to 0$ as $n \to \infty$.*

Now we can show Theorem 1.2.

Proof of Theorem 1.2. We set

$$L = \limsup_{|a|\to 1^-} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu}.$$

Let $K : \mathcal{B}^\alpha \to \mathcal{B}^\mu$ be any compact operator, $a \in \mathbb{D}$ fixed and define

$$f_a(z) = \frac{1}{\alpha 2^\alpha}\sigma_a(z), \quad (z \in \mathbb{D}).$$

Then f_a goes to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1^-$, $\|f_a\|_{\mathcal{B}^\alpha} \leq 1$ for all $a \in \mathbb{D}$ and

$$\|C_\phi - K\|^{\mathcal{B}^\alpha\to\mathcal{B}^\mu} \geq \|(C_\phi - K)\, f_a\|_{\mathcal{B}^\mu} \geq \frac{1}{\alpha 2^\alpha}\|\sigma_a \circ \phi\|_{\mathcal{B}^\mu} - \|K f_a\|_{\mathcal{B}^\mu}.$$

Hence, taking $\limsup_{|a|\to 1^-}$ and using Lemma 2.1, we obtain

$$\|C_\phi\|_e^{\mathcal{B}^\alpha\to\mathcal{B}^\mu} \geq \frac{1}{\alpha 2^\alpha}\limsup_{|a|\to 1^-}\|\sigma_a \circ \phi\|_{\mathcal{B}^\mu}. \tag{3}$$

Now, we go to show that there exists a constant $M_\alpha > 0$, depending only on α, such that

$$\|C_\phi\|_e^{\mathcal{B}^\alpha\to\mathcal{B}^\mu} \leq M_\alpha \limsup_{|a|\to 1^-}\|\sigma_a \circ \phi\|_{\mathcal{B}^\mu}.$$

Bearing this in mind, we define, for $r \in [0,1]$, the linear *dilation operator* $K_r : H(\mathbb{D}) \to H(\mathbb{D})$ by $K_r f = f_r$, where f_r, for each $f \in H(\mathbb{D})$, is given by $f_r(z) = f(rz)$. It is clear that if $f \in H(\mathbb{D})$ then $rf_r \to f$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1^-$. Also, the following statements hold:

(1) For $r \in [0,1)$, the operator K_r is compact on $\mathcal{B}^\alpha$,
(2) for each $r \in [0,1]$

$$\|K_r\|^{\mathcal{B}^\alpha\to\mathcal{B}^\alpha} \leq 1.$$

Hence, if we consider a sequence $\{r_n\} \subset (0,1)$ such that $r_n \to 1$ as $n \to \infty$ and define $K_n = K_{r_n}$, then for all $n \in \mathbb{N}$, the operator $C_\phi K_n$ is a compact from $\mathcal{B}^\alpha$ into $\mathcal{B}^\mu$ and by definition of the essential norm we have

$$\|C_\phi\|_e^{\mathcal{B}^\alpha\to\mathcal{B}^\mu} \leq \limsup_{n\to\infty}\|C_\phi - C_\phi K_n\|^{\mathcal{B}^\alpha\to\mathcal{B}^\mu}.$$

Thus, we have to show that

$$\limsup_{n\to\infty} \|C_\phi - C_\phi K_n\|^{\mathcal{B}^\alpha\to\mathcal{B}^\mu} \le M_\alpha L.$$

To see this, consider any $f \in \mathcal{B}^\alpha$ such that $\|f\|_{\mathcal{B}^\alpha} = 1$, then since

$$\|(C_\phi - C_\phi K_n) f\|_{\mathcal{B}^\mu} = |f(\phi(0)) - f(r_n\phi(0))| + \|(f - f_{r_n}) \circ \phi\|_\mu$$

and $|f(\phi(0)) - f(r_n\phi(0))| \to 0$ as $n \to \infty$, it is enough to show that

$$\limsup_{n\to\infty} \|(f - f_{r_n}) \circ \phi\|_\mu \le M_\alpha L.$$

Furthermore, since $r_n(f')_{r_n} \to f'$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, we have

$$\limsup_{n\to\infty} \sup_{|\phi(z)|\le r_N} \mu(z) \left|(f - f_{r_n})'(\phi(z))\right| \left|\phi'(z)\right| = 0,$$

where $N \in \mathbb{N}$ is large enough such that $r_n \ge \frac{1}{2}$ for all $n \ge N$. Hence we only have to show that

$$S := \limsup_{n\to\infty} \sup_{|\phi(z)|> r_N} \mu(z) \left|(f - f_{r_n})'(\phi(z))\right| \left|\phi'(z)\right| \le M_\alpha L.$$

Indeed, we write $S \le \limsup_{n\to\infty} (S_1 + S_2)$, where

$$S_1 = \sup_{|\phi(z)|> r_N} \mu(z) \left|f'(\phi(z))\right| \left|\phi'(z)\right| \quad\text{and}\quad S_2 = \sup_{|\phi(z)|> r_N} \mu(z) r_n \left|f'(r_n\phi(z))\right| \left|\phi'(z)\right|.$$

Then we have

$$\begin{aligned} S_1 &= \sup_{|\phi(z)|> r_N} \mu(z) \left|f'(\phi(z))\right| \left|\phi'(z)\right| \frac{v_\alpha(\phi(z)) \left|\sigma'_{\phi(z)}(\phi(z))\right|}{v_\alpha(\phi(z)) \left|\sigma'_{\phi(z)}(\phi(z))\right|} \\ &\le \frac{4}{\alpha} \|f\|_{\mathcal{B}^\alpha} \sup_{|\phi(z)|> r_N} \mu(z) \left|\sigma'_{\phi(z)}(\phi(z))\right| \left|\phi'(z)\right| \\ &\le \frac{4}{\alpha} \sup_{|\phi(z)|> r_N} \sup_{|a|> r_N} \mu(z) \left|\sigma'_a(\phi(z))\right| \left|\phi'(z)\right| \le \frac{4}{\alpha} \sup_{|a|> r_N} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu}, \end{aligned}$$

where we have used the relation (2) in the first inequality and the fact that $\|f\|_{\mathcal{B}^\alpha} \le 1$ in the second one. Taking limit as $N \to \infty$ we obtain

$$\limsup_{n\to\infty} S_1 \le \frac{4}{\alpha} L.$$

In similar way, we have

$$\begin{aligned} S_2 &\le \frac{4}{\alpha} \|f\|_{\mathcal{B}^\alpha} \sup_{|\phi(z)|> r_N} \mu(z) \left|\sigma'_{\phi(z)}(\phi(z))\right| \left|\phi'(z)\right| \frac{r_n v_\alpha(\phi(z))}{v_\alpha(r_n\phi(z))} \\ &\le \frac{4}{\alpha} \sup_{|a|> r_N} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu}, \end{aligned}$$

since $rv_\alpha(z) < v_\alpha(rz)$ for all $r \in (0,1)$ and all $z \in \mathbb{D}$. Therefore

$$\|C_\phi\|_e^{\mathcal{B}^\alpha \to \mathcal{B}^\mu} \leq \frac{8}{\alpha} \limsup_{|a|\to 1^-} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu}. \tag{4}$$

This completes the proof of the theorem. ■

As an immediate consequence of Theorem 1.2, we have the following corollary which generalize a result obtained recently by Malavé and Ramos-Fernández in [9] and extend a result due to Tjani in [14]. A similar result was found by Giménez, Malavé and Ramos-Fernández in [5], but for the composition operator $C_\phi : \mathcal{B} \to \mathcal{B}^\mu$, where the weight μ can be extended to non vanishing, complex valued holomorphic function that satisfy a reasonable geometric condition on the Euclidean disk $D(1,1)$.

Corollary 2.2. *The composition operator C_ϕ is compact from $\mathcal{B}^\alpha$ into $\mathcal{B}^\mu$ if and only if $\phi \in \mathcal{B}^\mu$ and*

$$\lim_{|a|\to 1^-} \|\sigma_a \circ \phi\|_\mu = 0. \tag{5}$$

References

[1] K. Attele. Toeplitz and Hankel operators on Bergman spaces, *Hokkaido Math. J.*, **21**, (1992), 279-293.

[2] R. E. Castillo, D. D. Clahane, J. F. Farías and J. C. Ramos-Fernández. Composition operators from logarithmic Bloch spaces to weighted Bloch spaces, *to appear in Appl. Math. Comput.*

[3] M. D. Contreras and A. G. Hernández-Díaz. Weighted composition operators in weighted Banach spaces of analytic functions. *J. Austral Math. Soc. (Serie A)*, **69**, (2000), 41-60.

[4] C. C. Cowen and B. D. MacCluer. *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995.

[5] J. Giménez, R. Malavé, J. Ramos-Fernández. Composition operators on μ-Bloch type spaces, *Rend. Circ. Mat. Palermo*, **59** 107-119 (2010).

[6] O. Hyvärinen, M. Kemppainen, M. Lindström, A. Rautio, and E. Saukko. The essential norm of weighted composition operators on weighted Banach spaces of analytic functions, *Integr. Equ. Oper. Theory*, **72**, (2012), 151-157.

[7] B. MacCluer and R. Zhao. Essential norms of weighted composition operators between Bloch-type spaces, *Rocky Mountain J. Math.*, **33**, (2003), 1437-1458.

[8] K. Madigan, A. Matheson. Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.*, **347**, (1995), 2679-2687.

[9] M. Malavé Ramírez, J. C. Ramos-Fernández. On a criterion for continuity and compactness of composition operators acting on α-Bloch spaces, *C. R. Math. Acad. Sci. Paris* (2013), http://dx.doi.org/10.1016/j.crma.2012.11.013.

[10] A. Montes-Rodríguez. The essential norm of composition operator on Bloch spaces, *Pacific J. Math.*, **188**, (1999), 339-351.

[11] A. Montes-Rodríguez. Weighted composition operators on weighted Banach spaces of analytic functions, *J. London Math. Soc. (2)*, **61**, (2000), 872-884.

[12] J. C. Ramos-Fernández. Composition operators between μ-Bloch spaces. *Extracta Math.* **26**, no. 1, (2011), 75-88.

[13] M. Tjani. *Compact composition operators on some Möbius invariant Banach space*, Ph. D. dissertation, Michigan State University, 1996.

[14] M. Tjani. Compact composition operators on Besov spaces. *Trans. Amer. Math. Soc.* **355**, no. 11, 4683-4698, (2003).
[15] H. Wulan. Compactness of composition operators on $BMOA$ and $VMOA$. *Sci. China Ser. A*, **50**, no. 7, 997-1004 (2007).
[16] H. Wulan, D. Zheng, and K. Zhu. Compact composition operators on BMO and the Bloch space, *Proc. Amer. Math. Soc.*, **137**, (2009), 3861-3868.
[17] J. Xiao. Composition operators associated with Bloch-type spaces, *Complex Variables Theory Appl.*, **46**, (2001), 109-121.
[18] K. Yang, Z. Zhou. Essential norm of the difference of composition operators on Bloch space, *Czechoslovak Math. J.*, **60(135)**, (2010), 1139-1152.
[19] R. Zhao. Essential norms of composition operators between Bloch type spaces, *Proc. Amer. Math. Soc.*, **138**, (2010), 2537-2546.

(Julio C. Ramos-Fernández) Departamento de Matemática, Universidad de Oriente, 6101 Cumaná, Edo. Sucre, Venezuela